Exercises Module 3

Exercise 3.1

We saw in exercise 2.6 that the speed is only radial and independent from θ (by symmetry):
$\boldsymbol{v} = \boldsymbol{v_r(r, z)}$

We consider a creeping flow (i.e. inertia forces are negligible in front of viscosity forces).

The Navier-Stokes equation, simplified (neglecting convective terms $\boldsymbol{v}.\boldsymbol{\nabla}(\rho\boldsymbol{v})$) and projected on $\overrightarrow{e_r}$ gives:

$$\underbrace{\cancel{\rho\frac{\partial v_r}{\partial t}}}_{\text{stationnary}} = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}}_{v_r \text{ independent from } \theta} + \frac{\partial^2 v_r}{\partial z^2} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}\right] + \cancel{\rho g_r} \qquad \textbf{Equation 1}$$

On the other hand, **the continuity equation** gives:

$$\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{\text{stationnary}} + \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \underbrace{\cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)}}_{v_\theta = 0} + \underbrace{\cancel{\frac{\partial}{\partial z}(\rho v_z)}}_{v_z = 0} = 0 \qquad \textbf{Equation 2}$$

Therefore:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0 \qquad \textbf{Equation 3}$$

Which allows to simplify further the Navier-Stokes equation (Equation 1):

$$\boldsymbol{0 = -\frac{\partial p}{\partial r} + \mu\frac{\partial^2 v_r}{\partial z^2}} \qquad \textbf{Equation 4}$$

We want the velocity profile $v_r(r, z)$

We will need 2 boundary conditions in z and one for the pressure:

B.C.1 : $v_r(r, b) = 0$ (no slip)

B.C.2 : $v_r(r, -b) = 0$ (no slip)

(alternatively) B.C.2 : $\left.\frac{dv_r}{dz}\right|_{z=0} = 0$ (symmetry condition)

B.C.3: $p(r_2, z) = p_2$ (e.g. atmospheric pressure)

To solve this differential equation, we have to examine the situation. We already know that the pressure only varies radially (when ignoring gravity) so $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{r})$

Moreover, from the continuity equation (Equation 3):

$$\frac{\partial}{\partial r}(rv_r) = 0$$

Therefore $\boldsymbol{rv_r} = \boldsymbol{f}(\boldsymbol{z})$

Replacing in equation 4:

$$\frac{dp}{dr} = \mu \frac{\partial^2 \left(\frac{f(z)}{r}\right)}{\partial z^2}$$

$$\frac{dp}{dr} = \frac{\mu}{r}\frac{d^2 f}{dz^2}$$

$$\boxed{\boldsymbol{\frac{dp}{dr} = \frac{\mu}{r}\frac{d^2(rv_r)}{dz^2}}}$$

<u>Note</u>**:** it is possible to find this differential equation directly by taking the equation we demonstrated in exercise 2.6, by neglecting the advective term (only leaving the viscosity term):

$$r\frac{dp}{dr} = \underbrace{\mu\frac{d^2(rv_r)}{dz^2}}_{\text{Viscous}} + \cancel{\rho v_r^2}\ \text{Advection}$$

Therefore:

$$r\frac{dp}{dr} = \mu\frac{d^2(rv_r)}{dz^2}$$

To solve this equation, we need to realize that the left-hand side term is a function of r only, whereas the right-hand side term is function of z only (as stated above $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{r})$ and $\boldsymbol{rv_r} = \boldsymbol{f}(\boldsymbol{z})$). Consequently, the only possibility for these two functions to be equal for all values of r and z is that they are **both equal to a constant**, independent from z and r:

$$r\frac{dp}{dr} = \mu\frac{d^2(rv_r)}{dz^2} = C_0$$

Therefore we have two equations:

$$\begin{cases} r\dfrac{dp}{dr} = C_0 \\ \mu\dfrac{d^2(rv_r)}{dz^2} = C_0 \end{cases}$$

This introduces another constant for which we will need another boundary condition. Fortunately, we can assume that $p(r_1) = p_1$. Indeed, similar to the laminar flow in the cylindrical pipe, we need to know the pressure drop per unit length to solve for the velocity profile. Here we can use a similar argument (we need to know pressure at both inlet and outlet to solve). Therefore we integrate the first differential equation between r_1 and r_2 assuming p_1 and p_2 are the respective pressures at these r's (in effect we are using the pressure B.C.s here).

$$\int_{p_1}^{p_2} dp = \int_{r_1}^{r_2} \frac{C_0}{r} dr$$

$$p_2 - p_1 = C_0 \ln\left(\frac{r_2}{r_1}\right)$$

$$\boldsymbol{C_0 = \frac{p_2 - p_1}{\ln\left(\frac{r_2}{r_1}\right)}}$$

Replacing in the second equation of the system:

$$\mu \frac{d^2(rv_r)}{dz^2} = \frac{p_2 - p_1}{\ln\left(\frac{r_2}{r_1}\right)}$$

Integrating twice over z:

$$\frac{d(rv_r)}{dz} = \frac{p_2 - p_1}{\mu \ln\left(\frac{r_2}{r_1}\right)} z + C_1$$

$$rv_r = \frac{p_2 - p_1}{2\mu \ln\left(\frac{r_2}{r_1}\right)} z^2 + C_1 z + C_2$$

Therefore

$$v_r(r,z) = \frac{p_2 - p_1}{2\mu r \ln\left(\frac{r_2}{r_1}\right)} z^2 + \frac{C_1 z}{r} + \frac{C_2}{r}$$

Since the flow is symmetric with respect to the plane $z = 0$,

$$v_r(r,z) = v_r(r,-z)$$

This implies that $\boldsymbol{C_1 = 0}$ (This can also be found using Alt.B.C.2: $\left.\frac{dv_r}{dz}\right|_{z=0} = 0$ (symmetry condition))

Moreover:

$$v_r(r,b) = 0$$

So

$$\frac{p_2 - p_1}{2\mu r \ln\left(\frac{r_2}{r_1}\right)} b^2 + \frac{C_2}{r} = 0$$

$$\boldsymbol{C_2 = -\frac{p_2 - p_1}{2\mu \ln\left(\frac{r_2}{r_1}\right)} b^2}$$

Finaly:

$$\boxed{\boldsymbol{v_r(r,z) = \frac{\Delta p b^2}{2\mu r \ln\left(\frac{r_2}{r_1}\right)}\left[1 - \frac{z^2}{b^2}\right]}}$$

Exercice 3.2

The Stokes flow equation:

$$\mu \boldsymbol{\nabla}^2 \boldsymbol{v} = \boldsymbol{\nabla} p$$

Moreover we have:

$$\begin{cases} v_r(r,\theta) = v_\infty \left(1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right)\cos\theta \\ v_\theta(r,\theta) = -v_\infty \left(1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right)\sin\theta \\ v_\phi = 0 \end{cases}$$

__Step 1__**: determination of the pressure profile**

So, in spherical coordinates, along r:

$$\frac{\partial p}{\partial r} = \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right]$$

$$\frac{\partial p}{\partial r} = \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(v_\infty\left(r^2 - \frac{3}{2}Rr + \frac{1}{2}\frac{R^3}{r}\right)\cos\theta\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(v_\infty\left(r^2 - \frac{3}{2}Rr + \frac{1}{2}\frac{R^3}{r}\right)\cos\theta\right)\right)\right]$$

$$\frac{\partial p}{\partial r} = \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(v_\infty\left(2r - \frac{3}{2}R - \frac{1}{2}\frac{R^3}{r^2}\right)\cos\theta\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta * v_\infty\left(1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right)(-\sin\theta)\right)\right]$$

$$\frac{\partial p}{\partial r} = \mu\left[\frac{1}{r^2}\left(v_\infty\left(2 + \left(\frac{R}{r}\right)^3\right)\cos\theta\right) + \frac{1}{r^2\sin\theta}\left(v_\infty\left(1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right) * (-2\cos\theta\sin\theta)\right)\right]$$

$$\frac{\partial p}{\partial r} = \mu\left[\cancel{\frac{2v_\infty}{r^2}\cos\theta} + \cancel{v_\infty\left(\frac{R}{r}\right)^3\cos\theta} - \cancel{\frac{2v_\infty}{r^2}\cos\theta} + 3v_\infty\left(\frac{R}{r^3}\right)\cos\theta - \cancel{v_\infty\left(\frac{R}{r}\right)^3\cos\theta}\right]$$

$$\boxed{\boldsymbol{\frac{\partial p}{\partial r} = 3v_\infty\left(\frac{R}{r^3}\right)\cos\theta = \frac{3v_\infty}{R^2}\left(\frac{R}{r}\right)^3\cos\theta}} \qquad \text{Equation 1}$$

And along θ:

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta)\right) + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

$$\begin{aligned}\frac{1}{r}\frac{\partial p}{\partial \theta} = \mu\Bigg[&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(-v_\infty\left(1-\frac{3}{4}\left(\frac{R}{r}\right)-\frac{1}{4}\left(\frac{R}{r}\right)^3\right)\sin\theta\right)\right)\\ &+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(-v_\infty\left(1-\frac{3}{4}\left(\frac{R}{r}\right)-\frac{1}{4}\left(\frac{R}{r}\right)^3\right)\sin^2\theta\right)\right)\\ &+\frac{2}{r^2}\frac{\partial}{\partial \theta}\left(v_\infty\left(1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)\cos\theta\right)\Bigg]\end{aligned}$$

$$\begin{aligned}\frac{1}{r}\frac{\partial p}{\partial \theta} = \mu\Bigg[&\frac{1}{r^2}\frac{\partial}{\partial r}\left(-v_\infty\left(\frac{3}{4}R+\frac{3}{4}\frac{R^3}{r^2}\right)\sin\theta\right)\\ &+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\cancel{\sin\theta}}\left(-v_\infty\left(1-\frac{3}{4}\left(\frac{R}{r}\right)-\frac{1}{4}\left(\frac{R}{r}\right)^3\right)*2\cancel{\sin\theta}\cos\theta\right)\right)\\ &+\frac{2}{r^2}\left(-v_\infty\left(1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)\sin\theta\right)\Bigg]\end{aligned}$$

$$\frac{\partial p}{\partial \theta} = \mu\left[v_\infty\frac{3}{\cancel{2}}\frac{R^{\cancel{3}}}{r^4}\sin\theta + 2v_\infty\left(\cancel{\frac{1}{r}}-\frac{3}{4}\left(\frac{R}{r^2}\right)-\cancel{\frac{1}{4}\frac{R^3}{r^4}}\right)\sin\theta - 2v_\infty\left(\cancel{\frac{1}{r}}-\frac{3}{2}\left(\frac{R}{r^2}\right)+\cancel{\frac{1}{2}\frac{R^3}{r^4}}\right)\sin\theta\right]$$

$$\boxed{\boldsymbol{\frac{\partial p}{\partial \theta} = \frac{3}{2}\mu v_\infty\left(\frac{R}{r^2}\right)\sin\theta = \frac{3}{2}\frac{\mu v_\infty}{R}\left(\frac{R}{r}\right)^2\sin\theta}}$$

Equation 2

From equation 1:

$$p(r,\theta) = \frac{3\mu v_\infty}{R^2}\cdot -\frac{1}{2}\frac{R^3}{r^2}\cos\theta + f(\theta)$$

$$p(r,\theta) = -\frac{3\mu v_\infty}{2R}\left(\frac{R}{r}\right)^2\cos\theta + f(\theta)$$

And equation 2 gives:

$$\frac{\partial p}{\partial \theta} = \frac{3\mu v_\infty}{2R}\left(\frac{R}{r}\right)^2 \sin\theta + \frac{df}{d\theta} = \frac{3}{2}\frac{\mu v_\infty}{R}\left(\frac{R}{r}\right)^2 \sin\theta$$

Therefore :

$$\frac{df}{d\theta} = 0 \implies f(\theta) = C_1$$

And:

$$p(r,\theta) = -\frac{3\mu v_\infty}{2R}\left(\frac{R}{r}\right)^2 \cos\theta + C_1$$

Boundary condition:

$$p(r \to \infty) = p_0 \Longrightarrow C_1 = p_0$$

Finally:

$$\boxed{\boldsymbol{p(r,\theta) = p_0 - \frac{3\mu v_\infty}{2R}\left(\frac{R}{r}\right)^2 \cos\theta}}$$

We now know the velocity and pressure profiles, so we are able to determine the forces applied on the sphere.

Step 2: pressure forces

The pressure at (r=R, θ=π), in “front” of the sphere is $\boldsymbol{p = p_0 + \frac{3\mu v_\infty}{2R}\left(\frac{R}{r}\right)^2}$

The pressure at (r=R, θ=0), at the “back” of the sphere, is $\boldsymbol{p = p_0 - \frac{3\mu v_\infty}{2R}\left(\frac{R}{r}\right)^2}$

Therefore, there is a pressure difference between the two sides of the sphere, which will induce a **net force on the sphere (this is a type of “form” drag, which we will discuss in later modules).**

If we place ourselves in cartesian coordinates, the flow is along the z direction, and will be symmetric with respect to the sphere along the x and y direction. This means the forces will compensate along x and y and **therefore the resulting friction forces have to be along the z direction.**

The force due to the pressure difference (form frag) on the z axis:

$$F_{p,z} = \int_{A_{sphere}} -p \; dA \, \boldsymbol{n} \cdot \boldsymbol{e_z} = \int_{A_{sphere}} -p \cos\theta \; dA$$

$$F_{p,z} = \int_0^{\pi} \int_0^{2\pi} -p(R,\theta) \cos\theta \, R^2 \sin\theta \; d\theta \, d\phi$$

$$F_{p,z} = -2\pi R^2 \int_0^{\pi} \left(p_0 - \frac{3\mu v_\infty}{2R} \cos\theta \right) \cos\theta \sin\theta \; d\theta$$

$$F_{p,z} = -2\pi R^2 \left[\int_0^{\pi} p_0 \cos\theta \sin\theta \, d\theta - \int_0^{\pi} \frac{3\mu v_\infty}{2R} \cos^2\theta \sin\theta \, d\theta \right]$$

$$F_{p,z} = -2\pi R^2 \; \cancel{\left[\frac{p_0}{2} \sin^2\theta \right]_0^{\pi}} + 3\pi R \mu v_\infty \left[-\frac{1}{3} \cos^3\theta \right]_0^{\pi}$$

$$F_{p,z} = 3\pi R \mu v_\infty * \frac{2}{3}$$

$$\boxed{\boldsymbol{F_{p,z} = 2\pi R \mu v_\infty}}$$

Step 3: the viscous force

Finally, we need to consider the shear force exerted by the friction of the viscous fluid on the surface of the sphere:

Again, the resulting force must be along the z axis:

$$F_{v,z} = \int_{A_{sphere}} -\tau_{r\theta}|_{r=R} (\boldsymbol{n} \cdot \boldsymbol{e_r}) dA \;\; \boldsymbol{e_\theta} \cdot \boldsymbol{e_z} = \int_{A_{sphere}} \tau_{r\theta}|_{r=R} \sin\theta \, dA$$

$$F_{v,z} = \int_0^{2\pi} \int_0^{\pi} \tau_{r\theta}|_{r=R} \sin\theta \, R^2 \sin\theta \; d\theta \, d\phi$$

Only this component of the stress tensor is non-zero at the surface of the sphere under the Stokes flow condition. This can be seen from the symmetry of the tangential flow or by pluggin the velocity profile into the stress tensor.

In spherical coordinates, for a Newtonian fluid, the stress tensor is expressed:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\theta\phi} & \tau_{\phi\phi} \end{bmatrix}$$

$$
= \begin{bmatrix} -2\mu \frac{\partial v_r}{\partial r} & -\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) & -\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right) \\ -\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) & -2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right) & -\mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) \\ -\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right) & -\mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) & -2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}\right) \end{bmatrix}
$$

Therefore, we have:

$$
F_{v,z} = \int_0^\pi \int_0^{2\pi} -\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)_{r=R} \sin\theta\, R^2 \sin\theta\ d\theta\, d\phi
$$

We have:

$$
\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) = \frac{\partial}{\partial r}\left(-v_\infty\left(\frac{1}{r}-\frac{3}{4}\left(\frac{R}{r^2}\right)-\frac{1}{4}\frac{R^3}{r^4}\right)\sin\theta\right) = -v_\infty \sin\theta\left(-\frac{1}{r^2}+\frac{3}{2}\frac{R}{r^3}+\frac{R^3}{r^5}\right)
$$

$$
\frac{\partial v_r}{\partial \theta} = \frac{\partial}{\partial \theta}\left(v_\infty\left(1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)\cos\theta\right) = -v_\infty\left(1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right)\sin\theta
$$

Therefore:

$$
F_{v,z} = \int_0^\pi \int_0^{2\pi} -\mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)_{r=R} \sin\theta\, R^2 \sin\theta\ d\theta\, d\phi
$$

$$
F_{v,z} = \int_0^\pi \int_0^{2\pi} \mu v_\infty \sin\theta\left(-\frac{1}{r}+\frac{3}{2}\frac{R}{r^2}+\frac{R^3}{r^4}+\frac{1}{r}-\frac{3}{2}\frac{R}{r^2}+\frac{1}{2}\frac{R^3}{r^4}\right)_{r=R} \sin\theta\, R^2 \sin\theta\ d\theta\, d\phi
$$

$$
F_{v,z} = \int_0^{2\pi} \int_0^\pi \mu v_\infty \sin\theta \frac{3}{2R} \sin\theta\, R^2 \sin\theta\ d\theta\, d\phi
$$

$$
F_{v,z} = 3\pi R\mu v_\infty \int_0^\pi \sin^3\theta\, d\theta
$$

$$
F_{v,z} = 3\pi R\mu v_\infty \int_0^\pi \sin^3\theta\, d\theta = 3\pi R\mu v_\infty \int_0^\pi \sin\theta\ \frac{1-\cos(2\theta)}{2}\, d\theta
$$

$$F_{v,z} = \frac{3}{2}\pi R\mu v_\infty \left[\int_0^\pi \sin\theta \, d\theta - \int_0^\pi \sin\theta \cos(2\theta) \, d\theta\right]$$

$$F_{v,z} = \frac{3}{2}\pi R\mu v_\infty \left[[-\cos\theta]_0^\pi - \int_0^\pi \frac{1}{2}(\sin(3\theta) - \sin\theta) \, d\theta\right]$$

$$F_{v,z} = \frac{3}{2}\pi R\mu v_\infty \left[-2 - \frac{1}{2}\left[\frac{1}{3}\cos(3\theta)\right]_0^\pi + \frac{1}{2}[\cos\theta]_0^\pi\right]$$

$$F_{v,z} = \frac{3}{2}\pi R\mu v_\infty \left(2 - \frac{1}{6} * (-2) - \frac{1}{2} * (-2)\right)$$

$$\boxed{\boldsymbol{F_{v,z} = 4\pi R\mu v_\infty}}$$

<u>Last step:</u> total force

Therefore, the total force applied on the sphere is:

$$\boxed{\boldsymbol{F_{total} = F_p + F_v = 6\,\pi R\mu v_\infty}}$$

This equation is known as the Stokes' law and gives the drag force of a fluid on a sphere.